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ISOPARAMETRIC SUBMANIFOLDS AND THEIR COXETER GROUPS

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0. Introduction

In the later 1930's Elie Cartan defined the notion of isoparametric functions on a space form N and began their study [4]–[7]. A smooth function $f\colon N \to R$ ($N = \mathscr{R}^{n+1}$, S^{n+1} or H^{n+1}), is isoparametric, if Δf and $|\nabla f|^2$ are functions of f. Among other things Cartan showed that the level hypersurfaces of f are parallel, and each has constant principal curvatures. And conversely, he showed that if M is a hypersurface of N with constant principal curvatures, then there is at least a *local* isoparametric function having M as a level. Cartan called such a hypersurface isoparametric. In the last ten years, many people carried forward this research [19, 25]. Finally around 1980, Münzner [18] completed the beautiful structure theory of isoparametric hypersurfaces in the spheres, and thereby reduced their classification to a (difficult!) algebraic problem. Many people subsequently made contributions to this classification problem including U. Abresch [1], D. Ferus, H. Karcher, H. F. Münzner [15], et al. While there has been considerable recent progress, it seems much remains to be done. By and large, the theory of isoparametric hypersurfaces has been a special subject by itself; however in recent years there have been applications to the theory of harmonic maps [12], and minimal submanifolds [14, 19, 23]. Recently, Eells [12] gave a definition of isoparametric maps for the purpose of constructing harmonic maps. S. Carter and A. West [3] gave a stronger definition of isoparametric maps from N^{n+m} to R^m; their purpose being to generalize Cartan's work to higher codimension. Using their definitions, they were able to show that there is a Coxeter group (i.e., a finite group generated by reflections) associated to each isoparametric map $f\colon N^{n+2} \to R^2$. However, they did not obtain a similar result for larger m. They were also unable to construct a *global* isoparametric map for a given isoparametric submanifold.

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However, this work of Münzner and Carter-West was instrumental in suggesting to me many of the ideas in this paper, whose main goal is to generalize Cartan's theory of isoparametric functions and hypersurfaces to a general theory of isoparametric maps and submanifolds.

In this paper, we make the following definitions:

Definition 1. A complete, connected, smooth n-dimensional submanifold M of N^{n+m} ($N = \mathbf{R}^{n+m}$, S^{n+m} or H^{n+m}) is called *isoparametric* if the normal bundle is flat and the principal curvatures of M in the directions of any parallel normal vector field are constant.

Definition 2. A smooth map $f = (f_{n+1}, \cdots, f_{n+m})\colon N^{n+m} \to R^m$ is called isoparametric if

(0) f has a regular point,

(1) $\nabla f_\alpha \cdot \nabla f_\beta$ and Δf_α are functions of f, for all α, β,

(2) $[\nabla f_\alpha, \nabla f_\beta]$ is a linear combination of $\nabla f_{n+1}, \cdots, \nabla f_{n+m}$ with coefficients being functions of f, for all α, β.

Our definition of isoparametric map is stronger than Eells' and weaker than Carter-West's. All three definitions agree with Cartan's when $m = 1$. We will show that a compact isoparametric submanifold M^n of R^{n+m} is isoparametric in R^{n+m} if and only if M^n is contained and isoparametric in a standard sphere of R^{n+m}. We also show that a noncompact complete isoparametric submanifold M^n of R^{n+m} is always a product $R^{m_1} \times M_1^{n-m_1}$, where M_1 is a compact isoparametric submanifold of R^{n+m-m_1} for some m_1. Therefore, in the following discussion, we will assume $N = R^{n+m}$ and M^n is a compact isoparametric submanifold of R^{n+m}. Since the set of all isoparametric hypersurfaces of S^{n+1} is the "same" as the set of all compact isoparametric submanifolds of R^{n+2} of codimension 2, Münzner's results gave a complete structure theory for them. There are many homogeneous and non-homogeneous codimension 2 isoparametric submanifolds in R^{n+2} [15, 20, 25]. There are also many examples from transformation group theory for higher codimension [22]. For example, a principal orbit of the adjoint action of a compact Lie group G on its Lie algebra $\mathfrak{G}$ is always isoparametric. More generally, if G acts on R^n orthogonally with a section [22], i.e. there is a submanifold F of R^n which meets every principal orbit orthogonally, then the principal orbit is isoparametric in R^n. In particular, these representations include the isotropy representations of symmetric spaces. Conversely, suppose G acts on R^n orthogonally and M is an orbit of G such that M is full and isoparametric in R^n. Then this section admits a section and M is a principal orbit of G [22]. Therefore all homogeneous isoparametric submanifolds are obtained from the principal orbits of linear orthogonal actions with sections. A simple example of a linear action with section is the action of $O(n)$ on the space S of real symmetric trace zero $n \times n$ matrices via conjugation. Then the canonical form in linear algebra for symmetric matrices implies that $F = \{a \in S | a \text{ is diagonal}\}$ is a section for this action. F is also the normal plane of the principal orbit at $\operatorname{diag}(\lambda_1, \cdots, \lambda_n)$, where λ_i's are distinct. The $O(n)$-invariant polynomial map f: $S \to R^{n-1}$ defined by $f(a) = (\operatorname{tr} a^2, \cdots, \operatorname{tr} a^n)$ is an isoparametric map. The principal orbit type is $O(n)/H$, where $H = \oplus_{i=1}^{n} Z_2$. The "Weyl group" $N(H)/H = S_n$ is a Coxeter group on R^{n-1}. One main conclusion of this paper is that these results still hold even if we replace the principal orbit by an arbitrary, not necessarily homogeneous, isoparametric submanifold M, and replace the orbit foliation by the family of all parallel submanifolds of M. Therefore, this example can be thought as a concrete model for the theory of isoparametric submanifolds.

A submanifold M^n of R^{n+m} is called *full* if M is not contained in any hyperplane of R^{n+m}. The proof of the following theorem is rather straight forward, and it will be given in §2.

Theorem A. *If f: $N^{n+m} \to R^m$ is isoparametric, then connected components of $f^{-1}(c)$ are isoparametric for regular value c, and all regular submanifolds of f are parallel.*

Conversely, given an isoparametric submanifold M^n of R^{n+m}, we can construct an isoparametric map f with M being a regular level of f. And there are many interesting facts that turn up in our proof of the converse.

Let M^n be full and isoparametric in R^{n+m}. We show that the holonomy group of $\nu(M)$ is trivial, in particular there is a global orthonormal parallel normal frame field e_α $(n+1 \leqslant \alpha \leqslant n+m)$ on M. And there exist p distributions E_i such that $TM = \oplus_{i=1}^{p} E_i$, and the second fundamental form in the direction of e_α restricted to E_i has only one eigenvalue n_i^α of multiplicity m_i, and $n_i = (n_i^{n+1}, \cdots, n_i^{n+m})$ are p distinct vectors in R^m. In §1, we prove that the group W generated by the p reflections R_i of R^m along $n_1, \cdots, n_p$ is an effective Coxeter group. We call W the Coxeter group associated to M. We also show that E_i is integrable, with leaf being an m_i-dimensional sphere of radius $1/|n_i|$. And the focal set of M at q is the union of p hyperplanes $l_i(q)$ in the affine normal plane $q + \nu(M)_q$. Each $l_i(q)$ is perpendicular to the leaf of E_i through q at its center. Moreover, we show that the group generated by the p reflections of the affine normal plane at q in $l_i(q)$ is isomorphic to W. Therefore, we have

Theorem B. *Let M^n be a full compact isoparametric submanifold of R^{n+m}, W the associated effective Coxeter group on R^m. Then*

(i) *W acts on M via diffeomorphisms; R_i corresponds to a diffeomorphism ϕ_i,* *where $\phi_i(q)$ is the antipodal point of q on the leaf (sphere) of E_i through q.*

(ii) *W acts on the affine normal plane $q + \nu(M)_q$ via rigid motions, R_i corresponds to the reflection T_i of $q + \nu(M)_q$ in $l_i(q)$. Moreover, $T_i(q) = \phi_i(q)$.*

If M is the principal orbit of the isotropy representation of a symetric space G/K, then the associated Coxeter group in Theorem B for M is the Weyl group (hence is crystallographic) [16], $q + \nu(M)_q$ is a maximal Lie triple system, n_i is always proportional to some root, and m_i is the number of roots proportional to n_i. Geometrically, m_i is the difference of dimensions of the principal orbit and the subprincipal orbits [22]. So the following theorem can be thought as a generalization of Chevalley Restriction Theorem [26] to isoparametric submanifolds.

Theorem C. *Let M^n be full and isoparametric in R^{n+m}, W the associated Coxeter group, $q \in M$ a given point on M, and V the affine normal plane $q + \nu(M)_q$. If u: $V \to R$ is a W-invariant homogeneous polynomial of degree k, then u can be extended uniquely to a homogeneous degree k polynomial f on R^{n+m}, which is constant on M.*

Now we only need to use another theorem of Chevalley to obtain the converse of Theorem A.

Theorem (*Chevalley* [9]). *If W is an effective Coxeter group on R^m, then the ring of W-invariant polynomials on R^m is a polynomial ring on m generators $u_1, \cdots, u_m$.*

Applying Theorem C to these generators $u_1, \cdots, u_m$, we obtain

Theorem D. *Let M, W, q, and V be as in Theorem C, and let $u_1, \cdots, u_m$ be a set of generators of the W-invariant polynomials on V. Then $u = (u_1, \cdots, u_m)$ extends uniquely to an isoparametric polynomial map f: $R^{n+m} \to R^m$ having M as a regular level set. Moreover,*

(1) *each regular level set of f is connected,*

(2) *the focal set of M is the set of all critical points of f,*

(3) *$V \cap M = Wq$,*

(4) *$f(R^{n+m}) = u(V)$,*

(5) *for $v \in V$, $f(v)$ is a regular value if and only if v is W-regular. We call f the Cartan polynomial map for M corresponding to the u_i.*

The proof of Theorem C and D are given in §3.

We also obtain a structure theory for isoparametric submanifolds. An isoparametric submanifold M^n of R^{n+m} is called irreducible if M cannot be written as a product of two lower dimensional isoparametric submanifolds. We show that every isoparametric submanifold can be written uniquely up to permutation as a product of irreducible ones. Moreover, M is irreducible if and only if the associated Coxeter group is irreducible. Therefore it suffices to classify all irreducible ones. It is obvious that if $f: R^{n+m} \to R^m$ is isoparametric and $\phi: R^m \to R^m$ is a diffeomorphism, then $\phi \circ f$ is also isoparametric. However, if we fix a set of generators for each Coxeter group, then the polynomial isoparametric map obtained in Theorem D is unique. The proof of Theorem D also provides an algebraic system of equations (completely determined by the Coxeter group W and the multiplicities m_i), of which f is a solution. Moreover, given any irreducible Coxeter group W on R^m with the positive root system $\Delta_+ = \{r_1, \cdots, r_p\}$ and positive integers m_i associated to each positive root r_i (such that $m_i = m_j$ if r_i and r_j are in the same orbit of W) the solutions of the algebraic system associated to W and m_i are isoparametric maps from R^{n+m} to R^m, where $n = \sum_{i=1}^p m_i$.

Let q_0 be a fixed point in an isoparametric submanifold M^n of R^{n+m}, and $q = q_0 + v \in q_0 + \nu(M)_{q_0}$. Then there is a unique parallel normal field V on M such that $V(q_0) = v$, and the parallel set through q, $M_q = \{x + V(x) | x \in M\}$, is *always* a smooth submanifold. Moreover, $\{M_q\}$ defines a singular foliation of R^{n+m}, which has M as one regular leaf, and all other regular leaves are also isoparametric. If M is a principal orbit of a linear action ρ with a section on R^{n+m}, then this singular foliation is the orbit foliation of ρ. To describe the general case in more detail, let U be the fundamental region of the Coxeter group W action on $q_0 + \nu(M)_{q_0}$. Then $\overline{U}$ is a simplicial cone, and $\overline{U}$ is the leaf space. If $q \in \mathrm{Int}(\sigma)$, where σ is a k-dimensional simplex of $\overline{U}$ and σ is open in $\bigcap_{j=1}^r l_{i_j}$, then the leaf M_q through q has dimension $n - \sum_{j=1}^r m_{i_j}$. Since we construct this singular foliation of R^{n+m} from M by a purely geometric construction, the geometry of this foliation is part of the geometry of M. There are also applications to the theory of harmonic maps and minimal submanifolds just as in the homogeneous case [14, 17, 21], even though in general there is no Lie group action around. We show in §4 that the Gauss map for each leaf is harmonic. We also prove that for each k-dimensional simplex σ as above, there is a point $q_k \in \sigma \cap S^{m-1}$ such that the leaf through q_k is a minimal submanifold of S^{n+m-1} of dimension $n - \sum_{j=1}^r m_{i_j}$.

So it is rather clear that isoparametric submanifolds and their singular foliations can be viewed as geometric generalizations of the principal orbits and the orbit foliations of linear actions with section. By studying the geometry of isoparametric submanifolds, we obtain a more clear geometric picture of these homogeneous spaces and their actions. It seems that Riemannian geometry also gives a different method to study invariant theory. For example, as a consequence of Theorem C, we see that if G acts on R^n orthogonally with a section then the ring of G-invariant polynomials on R^n is a polynomial ring.

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Added in revision. Professor G. Schwartz showed us a preprint of a paper by J. Dadok titled "polar coordinates induced by actions of compact Lie groups" [11], and pointed out to us that Dadok's definition of polar representation is equivalent to our definition of a linear action with a section. Moreover, Dadok classified polar representations in this paper and using his classification he showed that given any polar representation $\rho\colon G \to O(n)$ there is a symmetric space G/K and an isometry $A\colon R^n \to T(G/K)_{eK}$ such that the orbits of ρ are mapped under A to the orbits of $\mathrm{Ad}(K)$. We have also since noted a paper by L. Conlon [10], which has many of Dadok's results. Therefore, all the homogeneous isoparametric submanifolds are obtained from the principal orbits of the isotropy representations of symmetric spaces.

Some further progress has been made in understanding the nature of isoparametric submanifolds. For example, we have shown that the associated Coxeter group W for an isoparametric submanifold must be crystallographic, i.e. W is a Weyl group, and that in each irreducible example at most two distinct integers occur as multiplicities. Moreover, the homology and cohomology of isoparametric submanifolds can be computed from their Dynkin diagrams and multiplicities. These results will be reported in a separate paper by W. Y. Hsiang, R. S. Palais, and the author. It is also shown by the author in another paper that the convexity theorem of Kostant can be generalized to isoparametric submanifold M^n in R^{n+m}, namely the image by orthogonal projection of M to a fixed normal plane $\nu(M)_{x_0}$ is the convex hull of the orbit of x_0 under the Weyl group of M.

1. Isoparametric submanifolds

In this section, we will study the geometry of isoparametric submanifolds and associate to such manifolds Coxeter groups.

To set notations, we will review briefly the local geometry of submanifolds of Euclidean space. Let M be an n-dimensional submanifold of R^{n+m}. We choose a local orthonormal frame field $e_1, \cdots, e_{n+m}$ on R^{n+m} such that $e_1, \cdots, e_n$ are tangent to M. Let X be the position vector of M, i.e., X is the inclusion map from M to R^{n+m}. Let $w_1, \cdots, w_{n+m}$ be the dual coframe. We can write

$$(1.1) \qquad dX = \sum_i w_i e_i, \qquad de_A = \sum_B w_{AB} e_B.$$

Henceforth, we shall agree on the index ranges

$$1 \leqslant i, j, k \leqslant n, \quad n+1 \leqslant \alpha, \beta, \gamma \leqslant n+m, \quad 1 \leqslant A, B, C \leqslant n+m.$$

The structure equations of R^{n+m} are

$$(1.2)\quad dw_A = \sum_B w_{AB} \wedge w_B, \quad w_{AB} + w_{BA} = 0, \quad dw_{AB} = \sum_C w_{AC} \wedge w_{CB}.$$

Restricting these forms to M, we have

$$(1.3)\quad \begin{aligned} & w_\alpha = 0, \quad w_{i\alpha} = \sum_j h_{ij}^\alpha w_j, \quad h_{ij}^\alpha = h_{ji}^\alpha, \\ & dw_i = \sum_j w_{ij} \wedge w_j, \qquad w_{ij} + w_{ji} = 0, \\ & dw_{ij} - \sum_k w_{ik} \wedge w_{kj} = -\Omega_{ij} = -\sum_\alpha w_{i\alpha} \wedge w_{j\alpha}, \end{aligned}$$

where (w_{ij}) is the Levi-Civita connection and Ω is the Riemann tensor of M. There is also an induced connection on the normal bundle $\nu(M)$, namely $De_\alpha = \sum_\beta w_{\alpha\beta} e_\beta$. Then $dw_{\alpha\beta} + \sum_\gamma w_{\alpha\gamma} \wedge w_{\gamma\beta} = -\Omega_{\alpha\beta}$, where $\Omega_{\alpha\beta} = \sum_i w_{i\alpha} \wedge w_{i\beta}$ is the normal curvature of M. We say that $\nu(M)$ is flat if the normal curvature is zero. A normal vector field ν is parallel if $D\nu = 0$.

Suppose $\nu(M)$ is flat, then locally we can choose a normal orthonormal frame field e_α on M such that $w_{\alpha\beta} = 0$. The two fundamental forms are

$$\mathrm{I} = \sum w_i^2, \qquad \mathrm{II} = \sum_{\substack{i,j \\ \alpha}} h_{ij}^\alpha w_i w_j e_\alpha = \sum_\alpha \mathrm{II}_\alpha e_\alpha.$$

The eigenvalues of II_α with respect to I are called the principal curvatures of M in the directions of e_α. And $H = \sum_\alpha H_\alpha e_\alpha$ is the mean curvature vector, where $H_\alpha = \sum_i h_{ii}^\alpha$ is the mean curvature in the direction e_α.

1.1. Proposition. *Suppose $M^n \subset R^{n+m}$ has flat normal bundle, then locally there exists an orthonormal frame field e_A such that e_α is parallel and the II_α are diagonalized simultaneously, i.e., $w_{\alpha\beta} = 0$ and $w_{i\alpha} = \lambda_i^\alpha w_i$.*

Proof. We have noted that there is a local orthonormal normal frame e_α such that $w_{\alpha\beta} = 0$. Then

$$\begin{aligned} 0 = dw_{\alpha\beta} &= \sum_k w_{\alpha k} \wedge w_{k\beta} \\ &= -\sum_{k,\, i<j} \left(h_{ki}^\alpha h_{kj}^\beta - h_{kj}^\alpha h_{ki}^\beta\right) w_i \wedge w_j. \end{aligned}$$

So $A_\alpha A_\beta = A_\beta A_\alpha$, where $A_\alpha = (h_{ij}^\alpha)$. Hence there is an orthonormal local tangent frame field $e_1, \cdots, e_n$ such that A_α is diagonal for all α, i.e. $h_{ij}^\alpha = \lambda_i^\alpha \delta_{ij}$, or equivalently $w_{i\alpha} = \lambda_i^\alpha w_i$. q.e.d.

It follows from Definition 1 that if M^n is isoparametric in S^{n+m}, then M^n is also isoparametric in R^{n+m+1}. Later in this section we will show that a compact isoparametric submanifold of R^{n+m} is contained and isoparametric in a suitable standard sphere. So we will concentrate on isoparametric submanifolds of Euclidean space.

We will derive in the following some geometric properties of an isoparametric submanifolds M in R^{n+m}. Since $\nu(M)$ is flat, Proposition 1 gives a frame field e_A such that $w_{\alpha\beta} = 0$ and $w_{i\alpha} = \lambda_i^\alpha w_i$, where λ_i^α are constant on M. So there exist uniquely eigen-distributions $E_1,\cdots,E_p$ for II with dimensions $m_1,\cdots,m_p$ respectively, such that $\mathrm{II}_\alpha|E_i$ has only one eigenvalue n_i^α with multiplicity m_i, and $n_i = (n_i^{n+1},\cdots,n_i^{n+m})$ are p distinct vectors in R^m. We arrange our indices so that $\{e_k|\sum_{i=1}^{j-1} m_i < k \leqslant \sum_{i=1}^{j} m_i\}$ is a base for E_j. Next, we consider the following normal vectors $v_i = \sum_\alpha n_i^\alpha e_\alpha$. The v_i's are independent of the choice of the parallel frame e_α. To see this, let $e_\alpha^* = \sum_\beta s_{\alpha\beta} e_\beta$ be another local parallel normal frame. Then $(s_{\alpha\beta})$ is a constant $m \times m$ orthogonal matrix, and

$$\lambda_i^{*\alpha} = \sum_\beta s_{\alpha\beta} \lambda_i^\beta.$$

So it follows that

$$\sum_\alpha \lambda_i^{*\alpha} e_\alpha^* = \sum_\alpha \lambda_i^\alpha e_\alpha.$$

We call these v_i's the *curvature normal vectors* of M. If M^n is isoparametric in R^{n+m}, then M is obviously also isoparametric in R^{n+m+1}. To avoid this redundancy, we make the following definition:

1.2. Definition. A submanifold M^n of R^{n+m} is *full* if M is not contained in any hyperplane of R^{n+m}.

1.3. Proposition. *An isoparametric submanifold M^n in R^{n+m} is full if and only if the curvature normals $v_1,\cdots,v_p$ span $\nu(M)$. In particular, if M^n is full and isoparametric in R^{n+m}, then $m \leqslant n$.*

Proof. It is obvious that $v_1,\cdots,v_p$ span $\nu(M)$ if and only if the rank of the $m \times p$ matrix $N = (n_1,\cdots,n_p)$ is m. Suppose M is contained in a hyperplane normal to $(1,0,\cdots,0)$ in R^{n+m}. Then we can choose $e_{n+1} = (1,0,\cdots,0) \in R^{n+m}$, so $\lambda_i^{n+1} = 0$ for all i, and $\mathrm{rank}(n_1,\cdots,n_p) \leqslant m - 1$. Conversely, if $\mathrm{rank}(n_1,\cdots,n_p) < m$, then there is a nonzero vector $c \in R^m$ such that $c \cdot n_i = 0$ for all $1 \leqslant i \leqslant p$. We claim $\sum_\alpha c_\alpha e_\alpha$ is a constant vector b in R^{n+m}. To see this, we calculate the differential of the map $\sum_\alpha c_\alpha e_\alpha$ on M:

$$d\Big(\sum_\alpha c_\alpha e_\alpha\Big) = \sum_\alpha c_\alpha \, de_\alpha = -\sum_{i,\alpha} c_\alpha \lambda_i^\alpha w_i e_i = -\sum c \cdot n_i \,\mathrm{id}_{E_i} = 0.$$

Then it follows that

$$d(X \cdot b) = dX \cdot b = \sum_i w_i e_i \cdot b = 0.$$

Hence $X \cdot b = a$ constant c_0, i.e. M is contained in a hyperplane. q.e.d.

Next, we study the holonomy of $\nu(M)$.

1.4. Proposition. *Let M^n be full and isoparametric in R^{n+m}, and $v_1, \cdots, v_p$ its curvature normal vectors. Then the holonomy group G of $\nu(M)$ is a subgroup of the permutation group S_p. Moreover, if $v_1, \cdots, v_p$ have distinct length, then $G = \{\mathrm{id}\}$.*

Proof. Let r: $[0,1] \to M$ be a closed curve in M with $r(0) = r(1) = q$. Suppose parallel translation of $e_\alpha(q)$ along r is given by $\sum_\beta a_{\alpha\beta} e_\beta(q)$, i.e. $A = (a_{\alpha\beta})$ is in G. From the definition of isoparametric, there exists a permutation σ of $1, 2, \cdots, p$ such that

$$\lambda^\alpha_{\sigma(i)} = \sum_\beta a_{\alpha\beta} \lambda^\beta_i.$$

Let $n_1, \cdots, n_p$ be as before, then we have $An_i = n_{\sigma(i)}$, i.e. $G \subset S_p$. If $n_1, \cdots, n_p$ have distinct length, then since A is orthogonal we conclude that $A = \mathrm{id}$. q.e.d.

Remark. In the end of §3, we will show that the holonomy of $\nu(M)$ is always trivial.

For the remainder of this paper until 3.5, we will make the following standing assumptions:

(1) M^n is always assumed to be full and isoparametric in R^{n+m} with the inclusion map X: $M \to R^{n+m}$, and the holonomy of $\nu(M)$ is trivial.

(2) e_i is a local orthonormal tangent frame field and e_α is a global orthonormal parallel normal frame field on M. Such that

$$w_{\alpha\beta} = 0, \qquad w_{i\alpha} = \lambda^\alpha_i w_i, \tag{1.4}$$

where λ^α_i are constant.

(3) Let $E_1, \cdots, E_p$ be the unique eigendistributions on M for II. We let m_i be the rank of E_i, and arrange our indices so that E_i is spanned by $\{e_k | \sum_{j=1}^{i-1} m_j < k \leqslant \sum_{j=1}^{i} m_j\}$. Obviously $\sum_{j=1}^{p} m_j = n$.

(4) Let

$$n^\alpha_1 = \lambda^\alpha_1 = \cdots = \lambda^\alpha_{m_1},$$

$$n^\alpha_p = \lambda^\alpha_{\mu_p} = \cdots = \lambda^\alpha_n, \qquad \mu_p = 1 + \sum_{j=1}^{p-1} m_j.$$

Then $n_1, \cdots, n_p$ are p distinct vectors in R^m. The vectors $v = \sum_\alpha n^\alpha_i e_\alpha$ are the curvature normal vectors of M.

For a given global orthonormal parallel normal frame e_α, we define a normal bundle map

$$Y: M^n \times R^m \to R^{n+m}, \qquad Y(q, z) = q + \sum_\alpha z_\alpha e_\alpha(q).$$

Geometrically, $Y(q+\mathbf{R}^m)=q+\nu(M)_q$ is the affine normal plane at q, we denote it by N_q; and $Y(M\times\{z\})$ is an n-dimensional submanifold parallel to M in R^{n+m} for almost all z. The differential of Y is

$$\begin{aligned}dY&=d\left(X+\sum_\alpha z_\alpha e_\alpha\right)=dX+\sum_\alpha z_\alpha de_\alpha+\sum_\alpha dz_\alpha e_\alpha\\&=\sum_{i=1}^{p}(1-z\cdot n_i)\mathrm{id}_{E_i}+\sum_\alpha dz_\alpha e_\alpha.\end{aligned}\tag{1.5}$$

So we have

1.5. Proposition. *The critical point set of Y is $M\times\Lambda$, where Λ is the union of p hyperplanes l_i in R^m defined by the linear equations $1=n_i\cdot z$ for $i=1,\cdots,p$.*

1.6. Definition. The *focal set* Σ of M in R^{n+m} is the set of all critical values of the normal bundle map Y.

Focal set is well defined and independent of the choice of e_α. To see this, we let π: $\nu(M)\to M$ be the normal bundle, and $\tilde{Y}$: $\nu(M)\to R^{n+m}$ the map defined by $\tilde{Y}(v)=\pi(v)+v$. Then we have a canonical isomorphism ϕ: $M\times R^m\to\nu(M)$ with $\phi(x,z)=\sum_\alpha z_\alpha e_\alpha(x)$, and $Y=\tilde{Y}\circ\phi$. Then the focal set of M is the set of all critical values of Y, i.e. the focal variety of M in the classical sense.

It is obvious that we have

1.7. Proposition. $\Sigma=\bigcup_{q\in M}\Sigma_q$, *when Σ_q is the union of the p hyperplanes $l_i(q)=Y(q\times l_i)$ in $N_q=q+\nu(M)$, where l_i is the hyperplane of R^m defined by $z\cdot n_i=1$.*

In order to understand the eigendistributions, we need some formulas for the Riemannian connection of M in terms of E_i. Using (1.4) and the structure equations, we have

$$\begin{aligned}dw_{i\alpha}&=d(\lambda_i^\alpha w_i)=\lambda_i^\alpha\,dw_i=\lambda_i^\alpha\sum_j w_{ij}\wedge w_j\\&=\sum_j w_{ij}\wedge w_{j\alpha}=\sum_j\lambda_j^\alpha w_{ij}\wedge w_j,\end{aligned}$$

so

$$\sum_j\left(\lambda_i^\alpha-\lambda_j^\alpha\right)w_{ij}\wedge w_j=0.$$

Suppose

$$w_{ij}=\sum_k r_{ijk}w_k,$$

then we have

$$\sum_{j,k}\left(\lambda_i^\alpha-\lambda_j^\alpha\right)r_{ijk}w_k\wedge w_j=0,$$

so

$$(1.6) \qquad (\lambda_i^\alpha - \lambda_j^\alpha) r_{ijk} = (\lambda_i^\alpha - \lambda_k^\alpha) r_{ikj}.$$

Therefore we have

1.8. Proposition. *Let* $w_{ij} = \Sigma_k r_{ijk} w_k$. *If* $e_i, e_k \in E_{i_1}$, $e_j \in E_{i_2}$, *and* $i_1 \neq i_2$, *then* $r_{ijk} = 0$.

Using this proposition, we can obtain the following results for the eigendistributions:

1.9. Theorem. *Suppose* M^n *is full and isoparametric in* R^{n+m}, *then for all* $1 \leqslant i \leqslant p$, *we have*

(i) E_i *is integrable.*

(ii) *If* $n_i \neq 0$, *then the leaf* L_i *of* E_i *through* $q \in M$ *is an* m_i*-dimensional standard sphere of radius* $1/|n_i|$ *with center* c_i. *Moreover* $\bar{q} + v_i(\bar{q})/|n_i|^2 = c_i$ *for all* $\bar{q} \in L_i$, *and* $l_i(q)$ *intersects the* $(m_i + 1)$*-dimensional plane spanned by* L_i *orthogonally at* c_i.

(iii) *If* $n_i = 0$, *then* L_i *is an* m_i*-dimensional plane of* R^{n+m}.

(iv) *If* M *is compact, then* $n_i \neq 0$ *for all* $1 \leqslant i \leqslant p$.

(v) *Let* T_i *denote the reflection of* $N_q = q + \nu(M)_q$ *in the hyperplane* $l_i(q)$ *if* $n_i \neq 0$, *and* $T_i = \mathrm{id}$ *if* $n_i = 0$. *Then* $T_i(q) \in M$.

Proof. For simplicity, we assume $i = 1$. E_1 is defined by the following 1-form equations on M:

$$w_k = 0, \qquad m_1 < k \leqslant n.$$

Using the structure equations and Proposition 1.8, we have

$$dw_k = \sum_{i=1}^{m_1} w_{ki} \wedge w_i = \sum_{i,j=1}^{m_1} r_{kij} w_j \wedge w_i = 0.$$

By Frobenius theorem, E_1 is integrable.

Let L_1 be the leaf of E_1 through q. Then the above calculation also shows that $w_{ik} = 0$ for $i \leqslant m_1$, $k > m_1$. To compute the curvature of L_1, we use (1.3)

$$\Omega_{ij} = \sum_{k > m_1} w_{ik} \wedge w_{kj} + \sum_\alpha w_{i\alpha} \wedge w_{j\alpha} = |n_1|^2 w_i \wedge w_j.$$

So L_1 has constant sectional curvature $|n_1|^2$. If $n_1 = 0$, then we have $w_{ik} = 0$, $w_{i\alpha} = 0$ for $k > m_1$, $n < \alpha \leqslant n + m$; i.e. L_1 is totally geodesic in R^{n+m}. So if $n_1 = 0$, then L_1 is an m_1-dimensional plane of R^{n+m}, which proves (iii) and (iv). To obtain (ii), we divide the proof into four steps.

Step 1. The map $X + (1/|n_1|^2)\Sigma_\alpha n_1^\alpha e_\alpha$ maps L_1 to a constant vector c_1 in R^{n+m}. In particular, $c_1 = \bar{q} + v_1(\bar{q})/|n_1|^2$ for all $\bar{q} \in L_1$.

This can be seen directly from the differential of this map. We have

$$d\left(X+\frac{\sum_\alpha n_1^\alpha e_\alpha}{|n_1|^2}\right)=dX+\sum_\alpha\frac{n_1^\alpha}{|n_1|^2}de_\alpha$$
$$=\sum_i w_ie_i-\sum_{i,\alpha}\frac{n_1^\alpha}{|n_1|^2}\lambda_i^\alpha w_ie_i$$
$$=\sum_{i=1}^p\left(1-\frac{n_1\cdot n_i}{|n_1|^2}\right)\mathrm{id}_{E_i},$$

which vanishes on E_1.

Step 2. $\xi_1=E_1+\nu(M)|L_1$ is a fixed linear (m_1+m)-subspace of R^{n+m}.

We define a map $g\colon L_1\to\mathrm{Gr}(m_1+m,n+m)$ by $g(x)=$ the (m_1+m)-plane $E_1(M)_x\oplus\nu(M)_x$. Then locally, we can write

$$g=e_1\wedge\cdots\wedge e_{m_1}\wedge e_{n+1}\cdots\wedge e_{n+m}.$$

Using (1.1), we have

$$dg=\sum_{\substack{i\leqslant m_1\\ k>m_1}}w_{ik}e_1\wedge\cdots\wedge \underset{i}{e_k}\wedge\cdots\wedge e_{m_1}\wedge e_{n+1}\wedge\cdots\wedge e_{n+m}$$
$$+\sum_{\substack{k>m_1\\ \alpha}}w_{\alpha k}e_1\wedge\cdots\wedge e_{m_1}\wedge e_{n+1}\wedge\cdots\wedge \underset{\alpha}{e_k}\cdots\wedge e_{n+m}.$$

But $w_{ik}=0$, $w_{\alpha k}=0$ on L_1 for $i\leqslant m_1$, $k>m_1$. So $dg=0$.

Step 3. L_1 is contained in the (m_1+m)-plane $V_1=c_1+\xi_1$. This follows from Steps 1 and 2.

Step 4. The second fundamental form II of L_1 in V_1 is $(\sum_1^{m_1}w_i^2)v_1$.

We have $w_{ik}=0$ on L_1 for $k>m_1$, so $\mathrm{II}\cdot e_k=0$. Suppose $\sum_\alpha z_\alpha e_\alpha$ is perpendicular to $v_1=\sum_\alpha n_1^\alpha e_\alpha$, i.e., $z\cdot n_1=0$, then $\mathrm{II}\cdot\sum_\alpha z_\alpha e_\alpha=+\sum_{i=1}^{m_1}z_\alpha w_{i\alpha}w_i=\sum_{i=1}^{m_1}(n_1\cdot z)w_i^2=0$. Similarly

$$\mathrm{II}\cdot\frac{v_1}{|n_1|}=\sum_1^{m_1}\left(n_1\cdot\frac{n_1}{|n_1|}\right)w_1^2=|n_1|\sum_1^{m_1}w_i^2.$$

Finally to prove (v), we note that $T_1(q)\in L_1\subset M$. q.e.d.

Because $T_i(q)$ is the antipode of q in the sphere L_i, q goes to $T_i(q)$ gives a well-defined involution on M, i.e. we have

1.10. Corollary. *Suppose $n_i\neq 0$, and $\phi_i\colon M\to M$ maps q to the antipodal point of q in L_i, where L_i is the leaf of E_i through q. Then $\phi_i^2=\mathrm{id}$. In particular, we have*

$$\phi_i=X+\frac{2}{|n_i|^2}\sum_\alpha n_i^\alpha e_\alpha$$

is a diffeomorphism.

If $n_i = 0$, we let $\phi_i = \mathrm{id}$.

Next we study the general parallel submanifold of M given by the normal bundle map Y of M.

1.11. Proposition. *Let X^* be the map from M to R^{n+m} given by $X^* = Y|M \times \{z\}$, i.e. $X^* = X + \sum_\alpha z_\alpha e_\alpha$. Then $M^* = X^*(M)$ is an n-dimensional immersed isoparametric submanifold of R^{n+m} if and only if $1 - z \cdot n_i \neq 0$ for all $1 \leqslant i \leqslant p$. Moreover, if $q^* = X^*(q)$, then M^* and M have the same normal plane and focal sets at q and q^* respectively.*

Proof. Using (1.5), we have

$$dX^* = \sum_{i=1}^{p} (1 - z \cdot n_i)\, \mathrm{id}_{E_i}.$$

So dX^* has rank n if and only if $1 - z \cdot n_i \neq 0$ for all $1 \leqslant i \leqslant p$.

We may choose the following local frame on M^*:

$$e_\alpha^* = e_\alpha, \qquad e_i^* = e_i, \tag{1.7}$$
$$w_i^* = \Big(1 - \sum_\alpha z_\alpha \lambda_i^\alpha\Big) w_i.$$

Then

$$w_{\alpha\beta}^* = de_\alpha^* \cdot e_\beta^* = w_{\alpha\beta} = 0,$$

$$w_{i\alpha}^* = de_i^* \cdot e_\alpha^* = w_{i\alpha} = \lambda_i^\alpha w_i = \frac{\lambda_i^\alpha}{1 - \sum_\beta z_\beta \lambda_i^\beta} w_i^*,$$

which implies that M^* is isoparametric, e_α^* is a global parallel normal frame on M^*, and

$$n_i^* = \frac{n_i}{1 - z \cdot n_i}. \tag{1.8}$$

Since we have

$$X^* + \sum_\alpha y_\alpha e_\alpha^* = \Big(X + \sum_\alpha z_\alpha e_\alpha\Big) + \sum_\alpha y_\alpha e_\alpha$$
$$= X + \sum_\alpha (y + z)_\alpha e_\alpha,$$

the focal sets at q and q^* are the same. q.e.d.

We call an M^* as in Proposition 1.11 a *parallel submanifold* of M. Using Corollary 1.10 and Proposition 1.11 we have

1.12. Proposition. *If $n_i \neq 0$, then $1 - 2(n_i \cdot n_j)/|n_i|^2$ never vanishes for any $1 \leqslant j \leqslant p$.*

By examining more carefully the involutive diffeomorphism ϕ_i, we obtain

1.13. Theorem. *Suppose $n_i \neq 0$. Let $\phi_i(q) = q + 2v_i(q)/|n_i|^2$ be the diffeomorphism associated to the distribution E_i. Then the following hold:*

(i) *There exists a permutation σ_i of $1, 2, \cdots, p$, such that $\sigma_i(i) = i$, and $E_j(\phi_i(q)) = E_{\sigma_i(j)}(q)$ for all $q \in M$. In particular, we have $m_j = m_{\sigma_i(j)}$.*

(ii) *Let $\bar{e}_\alpha$ denote the normal frame field defined by $\bar{e}_\alpha(\phi_i(q)) = e_\alpha(q)$. Then $\bar{e}_\alpha$ is again a global parallel normal frame on M, and*

$$\bar{e}_\alpha = \sum_\beta \left(\delta_{\alpha\beta} - 2\frac{n_i^\alpha n_i^\beta}{|n_i|^2} \right) e_\beta,$$

i.e., $\bar{e}_\alpha = S_i e_\alpha$, where S_i is the reflection of $\nu(M)_q$ along $v_i(q)$.

(iii)

$$v_{\sigma_i(j)}(q) = \left(1 - 2\frac{n_i \cdot n_{\sigma_i(j)}}{|n_i|^2} \right) v_j(\phi_i(q)).$$

(iv)

$$S_i(v_j(q)) = v_j(\phi_i(q)) \quad \text{for all } 1 \leqslant j \leqslant p.$$

(v) *Let R_i be the reflection of R^m along n_i. Then*

$$R_i(n_j) = \left(1 - 2\frac{n_i \cdot n_{\sigma_i(j)}}{|n_i|^2} \right)^{-1} n_{\sigma_i(j)}.$$

Proof. We will assume $i = 1$, and $q' = \phi_1(q)$. It follows from Proposition 1.11 that $TM_q = TM_{q'}$, and $E_1(q), \cdots, E_p(q)$ are eigenspaces of the second fundamental form II of M at q'. Hence there is a permutation σ (which may depend on q) of $1, 2, \cdots, p$ such that $E_{\sigma(j)}(q) = E_j(q')$. It follows from Proposition 1.11 that $\bar{e}_\alpha = e_\alpha \circ \phi_1$ is a global parallel normal frame on M, so $\bar{e}_\alpha$, e_α, differ by a constant $O(m)$ matrix. To determine this matrix, we compare $\bar{e}_\alpha(q')$ and $e_\alpha(q')$. Note that $e_\alpha(q')$ is the parallel translation of $e_\alpha(q)$ on M along a path γ joining q and q' with respect to the normal connection of $\nu(M)$. But parallel translation in $\nu(M)$ is independent of the path, so we can choose γ to lie in the leaf L_1 of E_1 through q. From Theorem 1.9, there is an $(m_1 + m)$-dimensional plane V_1 such that $L_1 \subset V_1$, and the normal bundle of L_1 in V_1 is equal to $\nu(M)|L_1$. Therefore, parallel translation of $e_\alpha(q)$ to q' along γ with respect to the normal connection of $\nu(M)$ is the same as with respect to the normal connection of L_1 in V_1. Then it is easily seen that $e_\alpha(q')$ is the reflection of $e_\alpha(q)$ in $\nu(M)_q$ along $v_1(q)$, i.e.,

$$\bar{e}_\alpha = S_1 e_\alpha = e_\alpha - 2\frac{v_1 \cdot e_\alpha}{|n_1|^2} v_1 = \sum_\beta \left(\delta_{\alpha\beta} - 2\frac{n_1^\alpha n_1^\beta}{|n_1|^2} \right) e_\beta.$$

So we have

$$\bar{n}_i^\alpha = \sum_\beta \left(\delta_{\alpha\beta} - 2\frac{n_1^\alpha n_1^\beta}{|n_1|^2} \right) n_i^\beta = n_i^\alpha - 2\frac{n_1 \cdot n_i}{|n_1|^2} n_1^\alpha,$$

$$\bar{n}_i = R_1(n_i).$$

For the corresponding curvature normal vectors we have

$$v_i(q') = \sum_\alpha \bar{n}_i^\alpha \bar{e}_\alpha(q') = \sum \bar{n}_i^\alpha e_\alpha(q)$$

$$= v_i(q) - 2\frac{v_1 \cdot v_i}{|n_1|^2} v_1(q).$$

On the other hand, $E_i(q') = E_{\sigma(i)}(q)$ and (1.8) imply that

$$\bar{n}_i = \frac{1}{1 - z \cdot n_{\sigma(i)}} n_{\sigma(i)}, \qquad z = \frac{2n_1}{|n_1|^2}.$$

Therefore we obtain (v). This also proves that σ is independent of q. The rest of the theorem follows. q.e.d.

According to (v) of Theorem 1.3 the p reflections $R_1, \cdots, R_p$ permute the corresponding p reflection hyperplanes, so the root system of the group W generated by $R_1, \cdots, R_p$ is finite. Then it follows from the basic theory of Coxeter groups [2, Proposition 4.1.3, p. 37] that W is a Coxeter group.

1.14. Theorem. *Let M^n be full and isoparametric in R^{n+m}, e_α a global parallel normal frame, and $v_i = \Sigma_\alpha n_i^\alpha e_\alpha$ the curvature normal vectors. Then the group W generated by the p reflections $R_1, \cdots, R_p$ along n_i in R^m is an effective Coxeter group. We call W the Coxeter group associated to M.*

1.15. Corollary. *Let M^n be full and isoparametric in R^{n+m}, M^* parallel to M. Then the Coxeter groups of M and M^* are the same.*

Proof. Using (1.8), W^* and W have the same root system. q.e.d.

Now we are ready to prove Theorem B.

1.16. Proof of Theorem B. The group generated by $S_1, \cdots, S_p$ is obviously isomorphic to W. So to prove (ii), it suffices to prove that $S_{i_1} \cdots S_{i_r} = \mathrm{id}$ is equivalent to $T_{i_1} \cdots T_{i_r} = \mathrm{id}$. We proceed as follows: Let $q' = T_{i_1} \cdots T_{i_r}(q)$. Applying Theorem 1.13(iii) and (iv) repeatedly, we obtain

$$v_j(q') = S_{i_1} \cdots S_{i_r}(v_j(q)), \qquad v_j(q') \text{ is proportional to } v_{\sigma(j)}(q),$$

where $\sigma = \sigma_{i_1} \cdots \sigma_{i_r}$. Let $z \in R^m$ such that $q' - q = \Sigma_\alpha z_\alpha e_\alpha$. Define $X^* = X + \Sigma_\alpha z_\alpha e_\alpha$. Then X^* is a diffeomorphism, because X^* is the composition of diffeomorphisms $\phi_{i_1}\phi_{i_r} \cdots \phi_{i_r}$. The calculation in Proposition 1.11 shows that

$$v_j(q') = \frac{v_{\sigma(j)}(q)}{1 - n_{\sigma(j)} \cdot z}.$$

Suppose $T_{i_1} \cdots T_{i_r} = \mathrm{id}$, then $q' = q$, $v_j(q') = v_j(q) = S_{i_1} \cdots S_{i_r}(v_j(q))$. Hence $S_{i_1} \cdots S_{i_r} = \mathrm{id}$. Conversely, if $S_{i_1} \cdots S_{i_r} = \mathrm{id}$, then

$$v_j(q') = v_j(q) = \frac{v_j(q)}{1 - n_j \cdot z},$$

which implies that $n_j \cdot z = 0$ for all $1 \leqslant j \leqslant p$. Because M is full, Proposition 1.3 implies that $z = 0$, so $T_{i_1} \cdots T_{i_r}(q) = q$. From Proposition 1.11 we have an open ball B centered at q in the affine normal plane $q + \nu(M)_q$ so that for any $\bar{q} = q + \sum_\alpha z_\alpha e_\alpha(q) \in B$, $\bar{X} = X + \sum_\alpha z_\alpha e_\alpha$ defines a full isoparametric submanifold $\bar{M} = \bar{X}(M)$ of R^{n+m}. Moreover, $\bar{q} + \nu(\bar{M})_{\bar{q}} = q + \nu(M)_q$, $l_i(\bar{q}) = l_i(q)$, and $\bar{n}_i$ is proportional to n_i. So q in the above argument can be replaced by $\bar{q} \in B$. Therefore we have proved that the affine transformation $T_{i_1} \cdots T_{i_r}$ is the identity on B and hence everywhere. Since $\phi_i(q) = T_i(q)$, (i) is a consequence of (ii).

1.17. Corollary. *There exists a vector* $a \in R^m$ *such that* $a \cdot n_i = 1$ *for all* $n_i \neq 0$.

Proof. The group G generated by $T_1, \cdots, T_p$ on the affine normal plane $N_q = q + \nu(M)_q$ is isomorphic to W, so in particular the order $|G|$ of G is finite. For a finite subgroup G of the affine group N_q, $c = 1/|G| \sum_{g \in G} g(q)$ is a fixed point of G. Suppose $c = q + \sum_\alpha a_\alpha e_\alpha(q)$. Then $T_i(c) = c$, i.e. $c \in l_i(q)$, which implies that $a \cdot n_i = 1$ for all $n_i \neq 0$. q.e.d.

Now suppose $n_i \neq 0$ for all $1 \leqslant i \leqslant p$, and $a \cdot n_i = 1$ for all i. We claim that the map $X + \sum_\alpha a_\alpha e_\alpha$ is a constant vector $c \in R^{n+m}$ on M, because

$$d\left(X + \sum_\alpha a_\alpha e_\alpha\right) = \sum_{i=1}^{p} (1 - a \cdot n_i)\mathrm{id}_{E_i} = 0.$$

In particular, we have $|X - c|^2 = |a|^2$, i.e., M is contained in the sphere of radius $|a|$ and center c in R^{n+m}. Therefore we have proved

1.18. Theorem. *Suppose* M^n *is full isoparametric in* R^{n+m}, *and* $n_i \neq 0$ *for all* $1 \leqslant i \leqslant p$. *Then there exist vectors* $a \in R^m$, $c \in R^{n+m}$, *such that M is contained in the sphere of radius* $|a|$ *and center at* c. *In particular M is compact and* $\bigcap_{i,q} l_i(q) = \{c\}$.

1.19. Corollary. *Suppose* M^n *is contained in a sphere of* R^{n+m} *centered at the origin, and M is full and isoparametric. Then there exists a vector* $a \in \mathbf{R}^m$ *such that*

(i) $X = -\sum_\alpha a_\alpha e_\alpha$.

(ii) *All the affine normal planes* $q + \nu(M)_q$ *pass through the origin of* R^{n+m}.

Theorems 1.9 and 1.18 imply that a full isoparametric submanifold is compact if and only if $n_i \neq 0$ for all $1 \leqslant i \leqslant p$. Next, we will discuss the case when some n_i is zero.

If M^n is full and isoparametric in R^{n+m}, then $M^n \times R^l$ is also full and isoparametric in R^{n+m+l} with one of the curvature normal vectors being zero. The following theorem states the converse is also true.

1.20. Theorem. *If M^n is full and isoparametric in R^{n+m}, and $n_1 = 0$, then there is a full isoparametric submanifold M_1 of R^{n+m-m_1} such that $M = M_1 \times R^{m_1}$.*

Proof. By Corollary 1.17, $a \cdot n_i = 1$ for all $2 \leqslant i \leqslant p$. Consider

$$X^* = X + \sum_\alpha a_\alpha e_\alpha.$$

Then Proposition 1.11 implies that

$$dX^* = \sum_i (1 - n_i \cdot a)\mathrm{id}_{E_i} = \mathrm{id}_{E_1},$$

and $M^* = X^*(M)$ is a flat totally geodesic submanifold of R^{n+m}. So M^* is an m-dimensional plane and $X^*\colon M \to M^*$ is a submersion. We claim that $\oplus_{i=2}^p E_i$ is integrable. For if $X^*(q) = q^*$ then $X^*(L_i(q)) = q^*$ for all $i \geqslant 2$, i.e., $L_i(q) \subset (X^*)^{-1}(q^*)$. Hence $(X^*)^{-1}(q^*)$ is the integral submanifold of $\oplus_{i=2}^p E_i$. Note that $\oplus_{i=2}^p E_i$ is also defined by

$$w_i = 0, \qquad i \leqslant m_1,$$

so

$$0 = dw_i = \sum_{k > m_1} w_{ik} \wedge w_k = \sum_{k, l > m_1} r_{ikl} w_l \wedge w_k.$$

Hence

(1.9) $$r_{ikl} = r_{ilk} \quad \text{for } i \leqslant m_1,\ k, l > m_1.$$

From (1.6), we have

(1.10) $$\lambda_k^\alpha r_{ikl} = \lambda_l^\alpha r_{ilk}.$$

If $e_k, e_l \in E_j$ for some $j \geqslant 2$, then Proposition 1.8 implies that $r_{ikl} = 0$. If e_k and e_l belong to two different eigenspaces, then (1.9) and (1.10) imply that $r_{ikl} = 0$. Therefore we have that

$$w_{ik} = 0, \quad w_{i\alpha} = 0, \qquad i \leqslant m_1, k > m_1$$

on M. Then the fundamental theorem for submanifolds of Euclidean space implies that $M = R^{m_1} \times \tilde{M}$, where $\tilde{M} = X^{*-1}(q^*)$ is a compact isoparametric submanifold of R^{n+m-m_1}. q.e.d.

Next we discuss the irreducibility of the associated Coxeter group of an isoparametric submanifold, which leads to a decomposition theorem for isoparametric submanifolds.

If $M_i^{n_i}$ is isoparametric in $R^{n_i+l_i}$ with Coxeter group W_i on $\mathbf{R}^{l_i}$, for $i = 1, 2$, then $M_1 \times M_2$ is isoparametric in $\mathbf{R}^{n_1+n_2+l_1+l_2}$ with Coxeter group $W_1 \times W_2$ on $\mathbf{R}^{l_1+l_2}$. The converse is also true.

1.21. Theorem. *Let M^n be a compact full isoparametric submanifold of $\mathbf{R}^{n+m}$, W the associated Coxeter group. Suppose $\mathbf{R}^m = \mathbf{R}^{k_1} \times \mathbf{R}^{k_2}$, and $W = W_1 \times W_2$, where W_i is a Coxeter group on R^{k_i} for $i = 1, 2$. Then there exist two lower-dimensional isoparametric submanifolds M_1, M_2, such that $M = M_1 \times M_2$.*

Proof. We may assume that $n_i \in \mathbf{R}^{k_1} \times 0$, $R_i \in W_1$ for $i \leqslant p_1$, and $n_j \in 0 \times \mathbf{R}^{k_2}$, $R_j \in W_2$ for $j > p_1$. Let $a \in \mathbf{R}^{k_1} \times 0$ such that $a \cdot n_i = 1$ for all $i \leqslant p_1$. Consider $X^* = X + \sum_\alpha a_\alpha e_\alpha$, then $dX^* = \sum_{j>p_1}^{p} \mathrm{id}_{E_j}$. So a similar argument as in Theorem 1.20 will show that $\oplus_{i \leqslant p_1} E_i$ is integrable, and M is the product of the leaf of $\oplus_{i \leqslant p_1} E_i$ and $X^*(M)$. q.e.d.

Therefore we make the following:

1.22. Definition. An isoparametric submanifold M^n of R^{n+m} is called *irreducible* if M is not the product of two lower-dimensional isoparametric submanifolds.

As a consequence of Theorem 1.20 and 1.21, we have

1.23. Proposition. *An isoparametric submanifold of Euclidean space is irreducible if and only if the associated Coxeter group is irreducible.*

1.24. Theorem. *Every isoparametric submanifold of Euclidean space can be written as the product of irreducible ones, and such decomposition is unique up to permutation.*

2. Isoparametric map

In this section, we will prove Theorem A.

If $m = 1$, Definition 2 reduces to that of an isoparametric function given by E. Cartan [4]. For $m \leqslant n$, S. Carter and A. West [3] gave another definition of isoparametric map as follows: $f\colon N^{n+m} \to R^m$ is isoparametric if $A \cup (*A)$ is closed under exterior differentiation and wedge product, where $*$ is the $*$-operator for the Riemannian metric on N^{n+m} and $A = f^*(\wedge^* R^m)$. They show that when $m = 2$, their definition is equivalent to ours. However, for general m, their definitions seem to require stronger conditions than necessary, because of the following standard equalities:

$$*d*(df_i) = \pm\Delta f_i, \qquad *(df_i \wedge *df_j) = \langle \nabla f_i, \nabla f_j \rangle,$$

$$*d*(df_i \wedge df_j) = (\Delta f_i)\, df_j - (\Delta f_j)\, df_i + \text{dual of } [\nabla f_i, \nabla f_j].$$

Suppose $f\colon N^{n+m} \to R^m$ is isoparametric, then we may assume that at any regular point of f, there is a local orthonormal frame field $e_1, \cdots, e_n$, $e_{n+1}, \cdots, e_{n+m}$ with dual coframe $w_1, \cdots, w_{n+m}$, such that

$$df_\alpha = \sum_\beta c_{\alpha\beta} w_\beta \tag{2.1}$$

with $\operatorname{rank}(c_{\alpha\beta}) = m$, and $c_{\alpha\beta}$ are functions of f, so

$$dc_{\alpha\beta} \equiv 0 \pmod{w_{n+1}, \cdots, w_{n+m}}.$$

Hereafter we agree on the same indices notion as in §1. It is obvious that $w_\alpha = 0$ defines the level submanifolds of f. Condition (2) implies that the normal distribution defined by $w_i = 0$ is completely integrable.

2.1. Proposition. *Let $f\colon N^{n+m} \to R^m$ be isoparametric, $c = f(q)$ a regular value, $M = f^{-1}(c)$, and F the leaf of the normal distribution through q. Then*

(i) *F is totally geodesic.*

(ii) *$\nu(M)$ is flat and has trivial holonomy group.*

Proof. Take the exterior differential of (2.1), and using the structure equations, we obtain

$$\sum_\beta dc_{\alpha\beta} \wedge w_\beta + \sum_{\beta i} c_{\alpha\beta} w_{\beta i} \wedge w_i + \sum_\gamma c_{\alpha\beta} w_{\beta\gamma} \wedge w_\gamma = 0. \tag{2.2}$$

Because $dc_{\alpha\beta} \equiv 0 \pmod{w_{n+1}, \cdots, w_{n+m}}$ and the coefficient of $w_i \wedge w_\gamma$ of (2.2) is zero, we obtain

$$\sum_\beta c_{\alpha\beta}\big(-w_{\beta i}(e_\gamma) + w_{\beta\gamma}(e_i)\big) = 0.$$

But $\operatorname{rank}(c_{\alpha\beta}) = m$, hence

$$w_{\beta\gamma}(e_i) = w_{\beta i}(e_\gamma). \tag{2.3}$$

From condition (2) of Definition 2, we have

$$\begin{aligned}[e_\alpha, e_\beta] &= \sum_\gamma u_{\alpha\beta\gamma} e_\gamma = \nabla_{e_\alpha} e_\beta - \nabla_{e_\beta} e_\alpha \\ &= \sum_i \big(w_{\beta i}(e_\alpha) - w_{\alpha i}(e_\beta)\big) e_i + \sum_\gamma \big(w_{\beta\gamma}(e_\alpha) - w_{\alpha r}(e_\beta)\big) e_\gamma.\end{aligned}$$

Hence $w_{\beta i}(e_\alpha) = w_{\alpha i}(e_\beta)$. Using (2.3) and (2.4), we have

$$\begin{aligned} w_{\beta i}(e_\alpha) &= w_{\beta\alpha}(e_i) \\ &= w_{\alpha i}(e_\beta) = w_{\alpha\beta}(e_i) = -w_{\beta\alpha}(e_i).\end{aligned}$$

So

$$w_{\alpha\beta}(e_i) = 0 \quad \text{and} \quad w_{\alpha i}(e_\beta) = 0, \tag{2.4}$$

i.e.,

$$w_{\alpha\beta} = 0 \quad \text{on } M, \qquad w_{i\alpha} = 0 \quad \text{on } F.$$

We note that e_α on M can be obtained by applying the Gram-Schmidt process to $\nabla f_{n+1}, \cdots, \nabla f_{n+m}$, so e_α is a global parallel normal frame on M, hence the holonomy of $\nu(M)$ is trivial. q.e.d.

If $\nu(M)$ is flat, then Proposition 1.1 states that there is a local orthonormal tangent frame such that II_α are diagonalized simultaneously for all α.

2.2. Proposition. *The same notation as in Proposition* 2.1. *Then we have*

(i) *The mean curvature vector* $H = \Sigma_\alpha H_\alpha e_\alpha$ *on M is parallel.*

(ii) *The principal curvatures of M in the direction of* e_α *are constant for all* α.

Proof. We choose a local frame field e_A as in Proposition 1.1 such that $w_{i\alpha} = \lambda_i^\alpha w_i$, and $w_{\alpha\beta} \equiv 0 \pmod{w_{n+1}, \cdots, w_{n+m}}$.

For a smooth function ϕ on N, the gradient, the Hessian, and the Laplacian of ϕ are given by

$$d\phi = \sum_A \phi_A w_A,$$

$$\sum_B \phi_{AB} w_B = d\phi_A + \sum_B \phi_B w_{BA},$$

$$\Delta\phi = \sum_A \phi_{AA}.$$

Then a direct computation using the given frame gives us

$$df_\alpha = \sum_\beta c_{\alpha\beta} w_\beta, \qquad \Delta f_\alpha = \sum_\beta dc_{\alpha\beta}(e_\beta) - \sum_\beta c_{\alpha\beta} H_\beta,$$

where H_β is the mean curvature of level submanifolds in the direction of e_β. Since Δf_α, $c_{\alpha\beta}$, are functions of f, $\Sigma_\beta c_{\alpha\beta} H_\beta$ is a function of f. However $\mathrm{rank}(c_{\alpha\beta}) = m$, so H_α's are functions of f, i.e. H_α's are constant on M. e_A is chosen so that $w_{\alpha\beta} = 0$, so (i) is proved.

To prove (ii), we use the same method as used by Nomizu [13] for codimension one. We assume $N = R^{n+m}$, the other two cases are similar. Let X be the position vector of M in R^{n+m}, then $dX = \Sigma_1 w_i e_i$. For given e_α, and constant $t \in \mathbf{R}$, we define a map on an open neighborhood U of M:

$$X^* = X + te_\alpha,$$

then

$$dX^* = \sum_i (1 - t\lambda_i^\alpha) w_i e_i.$$

Therefore $X^*(U) = U^*$ is an open neighborhood of a level submanifold of f for small t. Moreover, using the same calculation as in Proposition 1.11, we have

$$\lambda_i^{*\alpha} = \frac{\lambda_i^\alpha}{1 - t\lambda_i^\alpha}, \qquad H_\alpha^* = \sum_i \frac{\lambda_i^\alpha}{1 - t\lambda_i^\alpha} = \sum_{k=0}^{\infty} \left(\sum_i (\lambda_i^\alpha)^{k+1} \right) t^k. \tag{2.5}$$

The left-hand side of (2.5) depends on t alone because of (i). Therefore $\Sigma_i(\lambda_i^\alpha)^{k+1}$ is a function of t for all k, i.e. λ_i^α's are constant on M. q.e.d.

As a consequence of Propositions 2.1 and 2.2 we obtain Theorem A.

3. Construction of polynomial isoparametric maps

In this section, given an isoparametric submanifold M^n of $\mathbf{R}^{n+m}$, we will construct a polynomial isoparametric map on R^{n+m} which has M as a level submanifold. This construction is a generalization of the Chevalley Restriction Theorem [26]. We also obtain a structure theory for all isoparametric submanifolds.

Because of the decomposition Theorem 1.24, we may assume M^n is a compact, full, and irreducible isoparametric submanifold of R^{n+m}, and W is the associated irreducible Coxeter group on R^m. We will use the same notations as in §1.

Let Y: $M^n \times R^m \to R^{n+m}$ be the normal bundle map associated to the global parallel normal frame e_α as in §1. Then there is a small ball B centered at the origin of R^m such that $Y|M \times B$ is a local coordinate system for R^{n+m}. In particular, $z \cdot n_i < 1$ for all $z \in B$, $1 \leqslant i \leqslant p$. We denote $Y(M \times B)$ by $\mathcal{O}$. In fact, $\mathcal{O}$ is a tubular neighborhood of M in R^{n+m}. From Theorem 1.18 and Corollary 1.19, we may assume that M is contained in a sphere centered at the origin in R^{n+m}, and that there is a vector $a \in R^m$ such that $X = -\Sigma_\alpha a_\alpha e_\alpha$. Then

$$Y = X + \sum_\alpha z_\alpha e_\alpha = \sum_\alpha (z_\alpha - a_\alpha) e_\alpha.$$

We let $y = z - a$. Then y_α is a smooth function defined on the tubular neighborhood $\mathcal{O}$ of R^{n+m}.

3.1. Proposition. *Let u: $R^m \to R$ be a W-invariant smooth function, where W is the Coxeter group associated to a full isoparametric compact submanifold M of R^{n+m}. Then the map f: $\mathcal{O} \to R$ defined by $f(Y(q, z)) = u(z - a)$ is a smooth function, and $f|M$ is constant. We call this f the extension of the W-invariant function u.*

In order to construct a global isoparametric map for M, we need the following lemmas.

3.2. Lemma. *Suppose u: $R^m \to R$ is a W-invariant homogeneous polynomial of degree k, then the function*

$$\phi(y) = \sum_{i=1}^{p} m_i \frac{\nabla u(y) \cdot n_i}{y \cdot n_i}$$

is a W-invariant homogeneous polynomial of degree $k - 2$.

Proof. Since $u(R_i y) = u(y)$, $\nabla u(R_i y) = R_i(\nabla u(y))$. We claim that $\nabla u(y) \cdot n_i = 0$ if $y \cdot n_i = 0$. For if $y \cdot n_i = 0$, then $R_i(y) = y$, so $\nabla u(y) = R_i(\nabla u(y))$, i.e. $\nabla u(y) \cdot n_i = 0$. Therefore $\phi(y)$ is a homogeneous polynomial of degree $k - 2$. To check that ϕ is W-invariant, we calculate

$$\begin{aligned}\phi(R_i y) &= \sum_j m_j \frac{\nabla u(R_i y) \cdot n_j}{R_i y \cdot n_j} \\ &= \sum_j m_j \frac{R_i(\nabla u(y)) \cdot n_j}{R_i y \cdot n_j} \\ &= \sum_j m_j \frac{\nabla u(y) \cdot R_i n_j}{y \cdot R_i n_j}.\end{aligned}$$

Using Theorem 1.13(i) and (v), we are done.

3.3. Lemma. *Let u: $R^m \to R$ be a W-invariant homogeneous polynomial of degree k, f: $\mathcal{O} \to R$ its extension. Then*

(i) *Δf is the extension of a W-invariant homogeneous polynomial of degree $(k - 2)$ on R^m.*

(ii) *$|\nabla f|^2$ is the extension of a W-invariant homogeneous polynomial of degree $2(k - 1)$ on R^m.*

Proof. Using (1.5), we may choose a local frame field $e_A^* = e_A$ on $\mathcal{O} \subset R^{n+m}$, and the dual coframe is

$$w_j^* = (1 - z \cdot n_i) w_j, \quad \text{for } \sum_{r=1}^{i-1} m_r + 1 \leqslant j \leqslant \sum_{r=1}^{i} m_r, \qquad w_\alpha^* = dz_\alpha.$$

The Levi-Civita connection 1-form on $\mathcal{O}$ is $w_{AB}^* = w_{AB}$. Then we have

$$dy_\alpha = w_\alpha^*, \qquad \Delta y_\alpha = -\sum_{i=1}^{p} \frac{m_i n_i^\alpha}{1 - z \cdot n_i} = \sum_{i=1}^{p} \frac{m_i n_i^\alpha}{y \cdot n_i}.$$

Since $f = u(y_{n+1}, \cdots, y_{n+k})$, we have

$$df = \sum_\alpha u_\alpha w_\alpha^*, \qquad |\nabla f|^2 = |\overline{\nabla} u|^2,$$

$$\Delta f = \overline{\Delta} u + \sum_i m_i \frac{\overline{\nabla} u \cdot n_i}{y \cdot n_i},$$

where $\overline{\Delta}$, $\overline{\nabla}$, are the standard Laplacian and gradient on $\mathbf{R}^m$. (i) follows from Lemma 3.2. To prove (ii), we note that $\overline{\nabla} u(R_i y) = R_i(\overline{\nabla} u(y))$, so $|\overline{\nabla} u|^2$ is a W-invariant polynomial of degree $2(k - 1)$ on $\mathbf{R}^m$.

3.4. Proof of Theorem C. We prove this theorem on $\mathcal{O}$ by using induction on the degree k of u. The theorem is obvious for $k = 0$. Suppose it is true for all $l < k$. Given a degree k W-invariant homogeneous polynomial u on R^m, using Lemma 3.3(ii), $|df|^2$ is again the extension of a W-invariant homogeneous polynomial of degree $2k - 2$ on R^m. Applying Lemma 3.3(i) repeatedly, we have $\Delta^{k-1}(|df|^2)$ is the extension of a degree zero W-invariant polynomial on R^m, hence it is a constant. Therefore

$$0 = \Delta^k\left(|df|^2\right)$$
$$= \sum_{l=0}^{k} \sum_{\substack{p+p'=k-l \\ i,i_1\cdots i_l}} c_{l,p}\left(\Delta^p f\right)_{i,i_1\cdots i_l}\left(\Delta^{p'} f\right)_{i,i_1\cdots i_l},$$

where $c_{l,p}$ are constants depending on l and p. We claim that

$$\left(\Delta^p f\right)_{i,i_1\cdots i_l}\left(\Delta^{k-l-p} f\right)_{i,i_1\cdots i_l}, \qquad p' = k - l - p,$$

is zero if $l < k$. For we may assume that $p \geqslant k - l - p$, so $p \geqslant 1$. By Lemmas 3.3(i), $\Delta^p f$ is the extension of a degree $k - 2p$ W-invariant polynomial on R^m. By the induction hypothesis, $\Delta^p f$ is a homogeneous polynomial on $\mathcal{O} \subset R^{n+m}$ of degree $k - 2p$, hence all the partial derivatives of order bigger than $k - 2p$ will be zero. We have $l + 1 > k - 2p$ by assumption, so we obtain

$$0 = \sum_{i,i_1\cdots i_k} f^2_{i,i_1\cdots i_k},$$

i.e., $D^\alpha f = 0$ in O for $|\alpha| = k + 1$. This proves that f is a homogeneous polynomial of degree k in O. There is a unique polynomial extension to R^{n+m}, which we still denote by f.

3.5. Proof of Theorem D. By a theorem of Chevalley [9] the ring of W-invariant polynomials on R^m is a polynomial ring with m generators $u_1,\cdots,u_m$. Let $f_1,\cdots,f_m$ be their extended polynomials on R^{n+m}. Then because $u_1,\cdots,u_m$ are generators, $f = (f_1,\cdots,f_m)$ will automatically satisfy conditions (0) and (1) of Definition 2. Since $[\nabla y_\alpha, \nabla y_\beta] = 0$ and f is a function of y, condition (2) of Definition 2 is also satisfied. The rest of the theorem follows from the fact that $u_1,\cdots,u_m$ separate orbits of W and regular points of the map $u = (u_1,\cdots,u_m)$ are the W-regular points. q.e.d.

The above proof also gives us a constructive method for finding all compact irreducible isoparametric submanifolds of Euclidean space. To be specific, given an irreducible Coxeter group W on R^m, we denote the root system by Δ, i.e.,

$$\Delta = \{ a \mid a \text{ is a unit vector in } R^m \text{ such that the reflection in } R^m \text{ along } a \text{ is in } W \}.$$

Choose $t \in R^m$ with $t \cdot a \neq 0$ for all $a \in \Delta$, and let Δ^+ denote the set of positive roots relative to t, i.e. $\Delta^+ = \{a \in \Delta | (a, t) \geqslant 0\}$. Suppose $\Delta^+ = \{a_1, \cdots, a_p\}$. Next associate a positive integer m_i to each a_i so that $m_i = m_j$ if a_i and a_j are in the same W orbit. Set $n = \sum_i^p m_i$. Let $u_1, \cdots, u_m$ be a fixed set of generators for the ring of W-invariant polynomials on R^m, which can be chosen to be homogeneous of degree k_i. Then there are W-invariant polynomials V_i, Φ_i, U_{ij}, and ψ_{ijk} such that

$$\Delta u_i = V_i(u), \qquad \nabla u_i \cdot \nabla u_j = U_{ij}(u),$$

$$\sum_j m_j \frac{\nabla u_i \cdot a_j}{y \cdot a_j} = \Phi_i(u), \qquad [\nabla u_i, \nabla u_j] = \sum_k \psi_{ijk}(u) \nabla u_k.$$

Then any polynomial solution $f = (f_1, \cdots, f_m)$: $R^{n+m} \to R^m$, with f_i being homogeneous of degree k_i, of the following system is an isoparametric map:

$$\begin{aligned} \Delta f_i &= V_i(f) + \Phi_i(f), \\ \nabla f_i \cdot \nabla f_j &= U_{ij}(f), \\ [\nabla f_i, \nabla f_j] &= \sum_k \psi_{ijk}(f) \nabla f_k. \end{aligned} \tag{3.1}$$

Moreover, if M is any regular level submanifold of such an f, then the associated Coxeter group of M is W and the rank of the eigendistributions $E_1, \cdots, E_p$ of M are $m_1, \cdots, m_p$, respectively.

Since u_1 can be chosen to be $\sum_1^m x_1^2$, the extension f_1 is $\sum_A x_A^2$. So (3.1) is a system of equations for $(m - 1)$ functions. Because both the coefficients and the admissible solutions for the system (3.1) are homogeneous polynomials, the problem of classifying isoparametric submanifolds becomes a purely algebraic one.

However, it is still not known for which irreducible Coxeter group W and multiplicities m_i, (3.1) has polynomial solutions. By a remarkable result of Münzner [18] (that $p = 1, 2, 3, 4, 6$) W must be crystallographic if $m = 2$, and it is natural to conjecture that the two remaining noncrystallographic irreducible Coxeter groups do not arise from nonhomogeneous isoparametric submanifolds.

Similar results can be proved for the hyperbolic space $H^n \subset R^{n,1}$. Moreover because of the algebraic nature of this problem, the author believes that the study of isoparametric submanifolds in the flat pseudo-Riemannian space $\mathbf{R}^{k,l}$ will yield to techniques similar to those we have used for the Euclidean case.

So far, the results in §§1 and 3 are proved under the assumption that the holonomy of $\nu(M)$ is trivial. Now we will show that this assumption is automatically satisfied.

3.6. Proposition. *If M^n is full and isoparametric in R^{n+m}, then the holonomy of $\nu(M)$ is trivial.*

Proof. Given a nonzero vector $v_0 \in \nu(M)_{q_0}$, let M^* denote the set of all points in R^{n+m} of the form $q + v$, where v is the parallel translation of v_0 along some path in M, joining q_0 to q. If v_0 is small enough, then M^* is an immersed submanifold. By Proposition 1.4, M^* is a finite cover of M. Note that Proposition 1.11 is a local result, hence M^* is isoparametric with $n_i^* = n_i/(1 - z \cdot n_i)$, and $1 - z \cdot n_i \neq 0$ for all i. Moreover,

$$S = \left\{ z \in R^m \,\middle|\, |1 - z \cdot n_i|/|n_i| = |1 - z \cdot n_j|/|n_j| \text{ for some } i \neq j \text{ and } n_i \neq 0,\, n_j \neq 0. \right\}$$

is the union of finitely many hyperplanes of R^m. So there is a small $z \notin S$. Let $v_0 = \Sigma_\alpha z_\alpha e_\alpha(q_0)$, then $n_1^*, \cdots, n_p^*$ are distinct. Proposition 1.4 implies that $\nu(M^*)$ has trivial holonomy. By Theorem D, there is a Cartan polynomial map f having M^* as a regular level. However, from the construction of f, we see that M is an open subset of regular level of f. But M is also compact, hence M is a regular level of f. This proves that the holonomy of $\nu(M)$ is trivial, and M^* is diffeomorphic to M. q.e.d.

3.7. Corollary. *Let M^n be full and isoparametric in R^{n+m}, e_α a global parallel normal frame, and $v = q_0 + \Sigma_\alpha z_\alpha e_\alpha(q_0)$ a W-regular point. Then $X = X + \Sigma_\alpha z_\alpha e_\alpha$ maps M diffeomorphically onto the parallel submanifold M^* of M through v.*

Remark. The author would like to thank S. Carter and A. West for pointing out a gap in the original proof of Proposition 3.6.

Next, we give an example to demonstrate that the theory of isoparametric submanifolds in an arbitrary flat manifold can be rather different. Let N be the flat 2-dimensional Mobius strip:

$$N = \{(x, y)/x \in [0, 1], y \in R\}/(0, y) \sim (1, -y).$$

Then $f(x, y) = y^2$ is isoparametric on N (i.e., Δf and $|\nabla f|^2$ are functions of f), and 0 is the only singular value of f. Let $M_t = f^{-1}(t)$. Then the holonomy of $\nu(M_t)$ is $\{\text{id}\}$ for $t > 0$, and Z_2 for $t = 0$; M_t is isoparametric; M_t, $t > 0$, is a double cover of M_0; and the normal bundle map has no focal points.

4. Applications to variational problems

Isoparametric submanifolds provide many solutions to natural variational problems in Riemannian geometry. In particular, we are interested in the following well-known functionals:

(1) The energy functional E for maps $f\colon M \to N$ [10], i.e.

$$E(f) = \int_M |df|^2 d\text{vol}(M).$$

Critical points of E are by definition harmonic maps.

(2) The area functionals A_k for immersions f: $M^k \to N$, i.e. $A_k(f) = k$-dimensional volume of M with respect to the metric on M induced by f. A critical point of A_k is called a minimal immersion. The gradient of A_k at f is the mean curvature vector of $f(M)$ in N [8].

In this section, we will make the following assumptions:

Let X: $M^n \to R^{n+m}$ be a full and isoparametric submanifold, W the Coxeter group associated to M with multiplicities m_i. Suppose M is contained in the unit sphere of R^{n+m}, then by Corollary 1.19 there is a unit vector $a \in R^m$ such that $X = -\Sigma_\alpha a_\alpha e_\alpha$. Moreover, the affine normal plane $N_q = q + \nu(M)_q$ is a linear m-plane of R^{n+m} and W acts on N_q orthogonally. Let U be the fundamental region of W on N_q containing q. Then U is a simplicial cone with m-faces [2, Chapter 4]. Using notation as in §1, we have

$$U = \left\{ q + \sum_\alpha z_\alpha e_\alpha(q) \middle| z \cdot n_i < 1 \text{ for all } 1 \leqslant i \leqslant p \right\},$$

$y = z - a$ is a linear coordinate system on N_q, and $y \cdot n_i = z \cdot n_i - 1$.

4.1. Theorem. *Suppose* $z \cdot n_i = 1$ *for* $i = i_1, \cdots, i_s$ $z \cdot n_i < 1$ *otherwise. Let* $X' = X + \Sigma_\alpha z_\alpha e_\alpha$, $M' = X'(M)$. *Then the following hold*:

(0) M' *is a submanifold of* R^{n+m} *with dimension* $n - \Sigma_{r=1}^s m_{i_r}$.

(i) *If* $z = n_i/|n_i|^2$, *then* M *is a* S^{m_i} *sphere bundle over* M'.

(ii) *Let* $q' = X'(q)$. *Then* $\nu(M')_{q'} = \nu(M)_q \oplus_{r=1}^s E_{i_r}(q)$, *and the mean curvature vector of* M' *at* q' *is*

$$H'(q) = \sum_\alpha H'_\alpha e_\alpha(q),$$

where

$$H'_\alpha = \sum_{j \neq i_1, \cdots, i_s} \frac{m_j n_j^\alpha}{1 - z \cdot n_j}.$$

(iii) $H'(q') \cdot v_{i_r}(q) = 0$ *for all* $1 \leqslant r \leqslant s$. *In particular, we have the identities*

$$\sum_{j \neq i_1, \cdots, i_s} \frac{m_j n_j \cdot n_{i_r}}{1 - z \cdot n_j} = 0 \quad \textit{for all } 1 \leqslant r \leqslant s.$$

Proof. $dX' = \Sigma_{i=1}^p (1 - z \cdot n_i) \mathrm{id}_{E_i}$ implies that dX' has constant rank $n - \Sigma_{r=1}^s m_{i_r}$, and $X'(L_i(q)) = X'(q)$ for all $i = i_1, \cdots, i_r$, which prove (0), (i) and the first part of (ii). It follows from the calculation in Proposition 1.11 that

$$H'(q) \cdot e_\alpha(q) = H'_\alpha = \sum_{j \neq i_1, \cdots, i_s} \frac{m_j n_j^\alpha}{1 - z \cdot n_j},$$

$$H'(q') \cdot e_i(q) = \sum_{j \neq i_1, \cdots, k_s} \frac{m_j \gamma_{jij}}{1 - z \cdot n_j} \quad \text{for } e_i(q) \in \bigoplus_{r=1}^s E_{i_r}(q),$$

where $w_{ij} = \Sigma_k \gamma_{ijk} w_k$.

By Proposition 1.8, we have $\gamma_{jij} = 0$, which proves the second part of (ii). It remains only to show that $H'(q') \cdot v_{i_r}(q) = 0$. It suffices to show that $H'(q') \cdot v$ is a constant for all unit vector $v \in E_{i_r}(q) \oplus \mathbf{R}v_{i_r}(q)$. To prove this, we note that from Theorem 1.9 $v_{i_r}/|n_{i_r}|$ defines a diffeomorphism from the leaf L_{i_r} of E_{i_r} to the unit sphere of $E_{i_r}(q) \oplus \mathbf{R}v_{i_r}(q)$, and the principal curvatures of M' in these directions can be calculated as follows:

$$\left(de_i', \frac{v_{i_r}}{|n_{i_r}|}\right) = \left(de_k, \frac{v_{i_r}}{|n_{i_r}|}\right) = \frac{n_{i_r} \cdot n_k}{|n_{i_r}|} w_k,$$

$$H'(q') \cdot \frac{v_{i_r}(\bar{q})}{|n_{i_r}|} = \frac{1}{|n_{i_r}|} \sum_{k \neq i_1, \cdots, i_s} \frac{m_k n_{i_r} \cdot n_k}{1 - z \cdot n_k}. \qquad \text{q.e.d.}$$

Applying the above theorem (ii) to the harmonic maps, we obtain

4.2. Theorem. *Let f: $R^{n+m} \to R^m$ be a full and isoparametric map, $c \in R^k$, then*

(i) *$f^{-1}(c)$ is always a submanifold of R^{n+m}.*

(ii) *The Gauss map of $M = f^{-1}(c)$ is a harmonic map from M to the appropriate Grassmann manifold.*

Proof. A theorem of Ruh and Vilms [24] states that if the mean curvature vector of a submanifold M^k of R^n is parallel, then the Gauss map g: $M \to \mathrm{Gr}(k, n)$ is harmonic. So to prove our theorem, it suffices to prove that the mean curvature vector of M' in Theorem 4.1 is parallel with respect to the normal connection of $\nu(M')$. To see this, we note that $H' = \sum H'_\alpha e'_\alpha$, where H'_α is constant, and $w_{\alpha j} = 0$ on M' for $e_j \in \oplus_{r=1}^{s} E_{i_r}$, so H' is parallel. q.e.d.

Using Theorem 4.1(iii), we also find many minimal submanifolds from isoparametric maps.

4.3. Theorem. *Let X: $M^n \to R^{n+m}$ be full and isoparametric, W the Coxeter group of M on R^m, and $U = \{y \in R^m | y \cdot n_i < 0\}$ the fundamental region of W on R^m. Suppose M is contained in the unit sphere of R^{n+m}, i.e., there is a unit vector a such that $X = -\sum_\alpha a_\alpha e_\alpha$. Then the following hold:*

(i) *$\Gamma = \overline{U} \cap S^{m-1}$ is a polyhedron of m faces in S^{m-1}, and each face is totally geodesic. In fact, S^{m-1} is invariant under W, and Γ is the closure of the fundamental region of the induced W-action on S^{m-1}.*

(ii) *If σ_i is the interior of an i-dimensional simplex of Γ, then there exist $n_{i_1}, \cdots, n_{i_s}$ such that $y \in \sigma_i$ if and only if $y \cdot n_{i_r} = 0$ for $r = 1, \cdots, s$ and $y \cdot n_i < 0$ otherwise. In particular, $M'_z = Y(M \times \{z\})$ are diffeomorphic for all $z \in a + \sigma_i$ with* dim *$M'_z = \nu_i$, where $\nu_i = n - \sum_{r=1}^{s} m_{i_r}$.*

(iii) *Let P_i: $\bar{\sigma}_i \to \mathbf{R}$ be defined by $P_i(y) = A_{\nu_i}(M'_{a+y})$. If $y_i \in \sigma_i$ is a critical point of P_i, then M'_{y_i+a} is minimal in S^{n+m-1}.*

(iv) *P_i assumes a maximum in σ_i. So there exists a ν_i-dimensional minimal submanifold of S^{n+m-1}. In particular, if $\sigma_0 = \{y_0\}$ is a vertex of Γ, then M'_{y_0+a} is minimal.*

Proof. (i) and (ii) are obvious. Suppose $y_i \in \sigma_i$ is a critical point of P_i. Let $y(t)$ be a curve in σ_i through y_i at $t = 0$, and $(dy/dt)(0) = b$. Then

$$(4.1) \qquad 0 = \frac{dP_i}{dt}(y(t))\Big|_{t=0} = H' \cdot \Big(\sum_\alpha b_\alpha e_\alpha\Big),$$

where H' denotes the mean curvature vector of M'_{y_i+a} in R^{n+m}. (4.1) is true for all $b \in R^m$ such that $b \cdot y_i = 0$ and $b \cdot n_{i_r} = 0$ for $1 \leqslant r \leqslant s$. By Theorem 4.1, we also have $H' \cdot v_{i_r} = 0$ for all $1 \leqslant r \leqslant s$, and $H' = \sum_\alpha H'_\alpha e_\alpha$. Hence H' is proportional to

$$\sum_\alpha (y_i)_\alpha e_\alpha = \sum_\alpha (-a + y_i + a)_\alpha e_\alpha = q + \sum (y_i + a)_\alpha e_\alpha(q),$$

which is the position vector of M'_{y_i+a}. So M'_{y_i+a} is minimal in S^{n+m-1}, which proves (iii). Now P_i is continuous in $\bar{\sigma}_i$, positive on σ_i and zero on the boundary of σ_i, hence P_i assumes a maximum in the interior. q.e.d.

Note that if $M^n \subset R^{n+m}$ is a principal orbit of a G-action on R^{n+m} with a section, then Theorems 4.2 and 4.3 are just applications of the symmetric criticality principal [21]. In particular, Theorem 4.3(ii) was proved by W. Y. Hsiang [17] for the homogeneous case. For general isoparametric submanifold M, even though there does not exist a group of isometries acting on M transitively, we still obtain the same results.

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